Gaussian beam quantum radar protocol

Lorenzo Maccone[1], Yi Zheng[2,3] and Changliang Ren[4]

1. *Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy*
2. *CAS Key Laboratory of Quantum Information,*
University of Science and Technology of China, Hefei 230026, People's Republic of China
3. *CAS Center for Excellence in Quantum Information and Quantum Physics,*
University of Science and Technology of China, Hefei 230026, People's Republic of China
4. *Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education,*
Key Laboratory for Matter Microstructure and Function of Hunan Province,
Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications,
Hunan Normal University, Changsha 410081, People's Republic of China

We present an entangled quantum radar protocol. It consists in scanning the sky with a thin Gaussian beam and measuring the travel time of the radiation reflected from the target, as in conventional radars. Here the Gaussian beam is composed of N photons entangled in the frequency degrees of freedom. We show that this provides a $\sqrt{N}$ quantum enhancement over the unentangled case, as is usual in quantum metrology.

In this paper we introduce a radar protocol that can achieve quantum enhanced ranging and target detection. Other quantum radar protocols [1–4] are typically based on the quantum illumination primitive [5, 6]: they can only discriminate the target presence or absence *at a predetermined specific point*, which implies that one must scan the whole 3d space in search of a target, which is impractical and time-consuming. Recently, a protocol that claimed to achieve quantum enhanced ranging capabilities was proposed in [7]. Unfortunately, its theoretical analysis was flawed as it was based on an incorrect optical transfer function [8]. This paper then, to our knowledge, presents the only known three dimensional quantum enhanced radar protocol that can give quantum-enhanced ranging to the target: we remind that radar stands for RAdio Detection And Ranging. A protocol for enhanced ranging in the idealized one-dimensional case was presented in [9], and this protocol is a sort of 3d extension of it. The analysis presented here is agnostic to the wavelength used, so the same protocol can be used also in the optical regime (lidar). It is also more practical than the previous protocol [7] since it does not require wideband radiation entangled in the wave vector $\vec{k}$ which would require large antennas or antenna arrays: the protocol presented here only employs Gaussian thin beams that can be produced with small antennas (or lasers). The beams are entangled only in the frequency degrees of freedom, which is more practical. There is no quantum enhancement in the transversal direction, although this feature can be added to our protocol, for example using the techniques presented in [10], namely by injecting squeezed vacuum in the modes orthogonal to the Gaussian mode used by the protocol, or with similar techniques.

As in the case of most other quantum metrology protocols [11–15], we show an enhancement in the precision of the order of $\sqrt{N}$, where N is the number of photons employed in the ranging procedure. Namely, we show that this protocol can achieve the Heisenberg bound in precision in the ideal noiseless situation. As usual, the situation becomes extremely more complicated in the presence of noise, such as loss of photons, but the usual general procedures and techniques to deal with noise can be applied also in this case [16–19], e.g. one can increase the robustness by reducing the entanglement (and hence the precision gain) [20].

The outline follows: we start in Sec. I by showing how one can consistently derive the correct transfer functions in quantum optics, based on the quantization of the electromagnetic (EM) field which is reviewed (to set the notation) in App. A. We then show in Sec. II how these techniques can be used to give a quantum description of the usual classical radar protocols. This is useful to show what are the ultimate bounds (due to quantum mechanics) that can be achieved by these protocols in the absence of entanglement. Finally, in Sec. III we introduce and analyze our proposed protocol, and show its $\sqrt{N}$ enhancement.

I. QUANTIZATION OF OPTICS THROUGH TRANSFER FUNCTIONS

In this section we review how optical transfer functions can be consistently quantized. The notation and the framework we employ is given in App. A. The linearity and shift-independence form of the Helmholtz equation (A3) implies that its solutions $U_\omega(\vec{r})$ can be shifted in space, e.g. along the z axis [21]:

$$U_\omega(\vec{r}') = \int d^2\vec{r}_t \, U_\omega(\vec{r}_t, z) \, h_\omega(\vec{r}_t\,', \vec{r}_t, z' - z) \,, \quad (1)$$

where the t index represents the two-dimensional transverse vector $\vec{r}_t = (x, y)$, and where $\vec{r} = (\vec{r}_t, z)$ and h_ω is the transfer function that takes the solution U_ω at the xy plane at position z and, with a convolution, moves it to the xy plane at position z' (so the left-hand-side is independent of z). This allows us to obtain the field at all positions starting from the boundary values of the

field in the plane xy at position z. Of course, the general solution of the field is given by (A1), where one must sum U_ω over all components ω. Indeed, replacing Eq. (1) into (A2), we obtain the whole field $A(t, \vec{r}')$ at position $\vec{r}' = (\vec{r}'_t, z')$ from a field on the plane xy at position z at time $t = 0$ (boundary conditions):

$$A^+(t, \vec{r}') = \int d\omega \, d^2\vec{r}_t U_\omega(\vec{r}_t, z) \, h_\omega(\vec{r}_t{}', \vec{r}_t, d) e^{-i\omega t}, \quad (2)$$

where, for simplicity of notation, we consider only the positive-energy component of the field A^+, namely only the first term in the integral of (A2).

This transfer function formalism is developed in the classical case but it can be transferred to the quantum case by first expressing the solutions $U_\omega(\vec{r})$ in terms of plane waves $e^{i\vec{\kappa}\cdot\vec{r}}$, and then associating to each plane wave an amplitude $a(\vec{\kappa})$ as done in the customary EM quantization (see App. A). Namely,

$$A^+(t, \vec{r}') = \int d^2\vec{r}_t \, d^3\vec{\kappa} \, h_{\omega_\kappa}(\vec{r}_t{}', \vec{r}_t, d) a(\vec{\kappa}) e^{-i(\omega_k t - \vec{\kappa}\cdot\vec{r})}, (3)$$

where $\vec{r} = (\vec{r}_t, z)$, $\vec{r}' = (\vec{r}'_t, z')$, and the integral over ω is contained in the integral over $\vec{\kappa}$, since $\omega_\kappa = c\kappa$. [More rigorously, the integral over ω comes from (2), whereas in the input field we are considering only the ω component $U_\omega(\vec{r})$, so we need to integrate only on the directions $\vec{\kappa}/\kappa$ as discussed below Eq. (A4).] For example, $\vec{r}$ may represent the object plane and $\vec{r}'$ the image plane in an imaging apparatus, whose transfer function is given by h_ω [22]. Eq. (3) is the main result of this section.

This is the field *operator*, so by itself it says nothing about the physics: operators in quantum mechanics only acquire values when applied to states, e.g. the probability $p(t, \vec{r}) \propto |\langle 0|A^+|\psi\rangle|^2$. Alternatively, we may be interested in other expectation values of the field in state $|\psi\rangle$. The field degrees of freedom (including its boundary conditions) are encoded into $|\psi\rangle$. E.g., for a single photon with $\psi(\vec{\kappa}) \propto \alpha(\vec{\kappa})$ [see Eq. (A10)], we have

$$\langle 0|A^+(t, \vec{r}')|\psi\rangle$$

$$= \int d^3\vec{\kappa} \, d^3\vec{\kappa}' \, d^2\vec{r}_t \, h_{\omega'} \, e^{-i(\omega' t - \vec{\kappa}'\cdot\vec{r}')} \alpha(\vec{\kappa}) \langle 0|a(\vec{\kappa}')a^\dagger(\vec{\kappa})|0\rangle$$

$$= \int d^3\vec{\kappa} \, d^2\vec{r}_t \, h_\omega \, e^{-i(\omega t - \vec{\kappa}\cdot\vec{r})} \, \alpha(\vec{\kappa})$$

$$= \int d\omega \, d^2\vec{r}_t \, h_\omega(\vec{r}_t{}', \vec{r}_t, d) \, U_\omega(\vec{r}) \,, \quad (4)$$

where we used the fact that integrating $\alpha(\vec{\kappa})e^{i\vec{\kappa}\cdot\vec{r}}$ over the directions of $\vec{\kappa}$, one obtains $U_\omega(\vec{r})$ with $\omega = c\kappa$, as is clear by the comparison between Eqs. (A2) and (A4). This result is what one would expect from (1) by integrating over ω both members.

Free field transfer function

The specific form of the function h_ω depends on what is present between the two xy planes at z and z', and on the approximations used. In the case of vacuum propagation with the Fresnel approximation, we get ([21], Eq. 4.1-14)

$$h_\omega(\vec{r}_t{}', \vec{r}_t, d) = \frac{i\kappa}{2\pi d} e^{-i\kappa(\vec{r}_t - \vec{r}_t{}')^2/d} \, e^{-i\kappa d} \,, \quad (5)$$

with $d = z' - z$ the distance between the two planes. While the Rayleigh-Sommerfeld diffraction can give better results in some cases, in the regimes we are interested, the Fresnel approximation that gives rise to (5) is sufficient to our aims. We will be using Eq. (5) in the following.

II. QUANTUM TREATMENT OF A CLASSICAL RADAR/LIDAR PROTOCOL

A radar/lidar works by scanning the sky with a directional beam and measuring the time it takes for it to be bounced back. The direction of the beam and the time of flight suffice to do a full 3d localization of the target. In this section we analyze a classical radar/lidar protocol using quantized light to show what are the ultimate bounds imposed by quantum mechanics to such classical (unentangled) protocols.

As directional beam, we consider a Gaussian beam. For simplicity we will consider the target as a perfectly (or partially) reflecting mirror orthogonal to the beam direction, of size larger than the beam waist at the target location. In this way we are guaranteed that the beam that returns to the antenna is still in a (possibly attenuated) Gaussian beam, see Fig. 1. The case in which the target is smaller than the beam waist should also not be too difficult: the returning beam will be a spherical wave originating at the target.
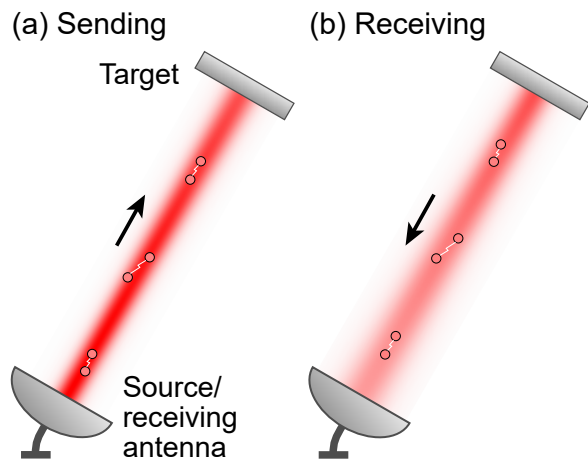

FIG. 1: Sketch of the quantum radar protocol. A Gaussian beam composed by frequency-entangled photons bounces off the target and returns to the sender's location. By measuring the average photon round-trip time, the sender can recover the target's position with quantum enhanced accuracy.

The Gaussian beam for every frequency ω has an amplitude $U_\omega(\vec{r}) = \varphi(\omega)G_\omega(\vec{r})$, where $\varphi(\omega)$ is the spectral

amplitude (the amplitude for each frequency ω in the light) and G_ω is ([21], Eq. 3.1-7)

$$G_\omega(\vec{r}) \propto \frac{1}{W(z)} e^{-\frac{\kappa r_t^2}{2z_0 W^2(z)}} e^{-i[\kappa z + \frac{\kappa r_t^2}{2z R(z)} - \arctan(z/z_0)]} \quad (6)$$

where $W(z) \equiv \sqrt{1 + z^2/z_0^2}$, $R(z) \equiv 1 + z_0^2/z^2$, z_0 is a (length) constant that, together with the direction of the z axis, fully specifies $G_\omega(\vec{r})$. It is possible to check that this solution for fixed z is propagated to an arbitrary z' through the transfer function (5). The field A is obtained by integrating U_ω over ω, as in Eq. (A1):

$$A(t, \vec{r}) = \int d\omega \, e^{-i\omega t} \, \varphi(\omega) \, G_\omega(\vec{r}) \,. \quad (7)$$

We now consider a single photon in a Gaussian beam[1]. Since the light intensity $|A(\vec{r})|^2$ at each point is directly proportional to the probability of finding the photon there (as discussed above) [21, 23], we can choose $\tilde{\psi}(\vec{r}) \propto A = \int d\omega \, \varphi(\omega) \, G_\omega(\vec{r})$, using (7) with $t = 0$ (because of the Heisenberg picture) and the proportionality constant chosen by the normalization condition. Namely, the photon wavepacket has probability amplitude proportional to $G_\omega(\vec{r})$ for each frequency ω, and the probability amplitude of having frequency ω is given by $\varphi(\omega)$. So the state is

$$|\psi\rangle = \int d^3\vec{\rho} \, \tilde{\psi}(\vec{\rho}) \, a^\dagger(\vec{\rho})|0\rangle = \int d^3\vec{\kappa}' \, \tilde{G}(\vec{\kappa}') \, a^\dagger(\vec{\kappa}')|0\rangle \,, \quad (8)$$

where $\tilde{G}(\vec{\kappa})$ is the Fourier transform of $\varphi(\omega)G_\omega(\vec{\rho})$, which clearly only contains the frequency ω (the amplitude φ is included in $\tilde{G}$). We can write the field at the image plane, i.e. the detector position $\vec{r}'$ in terms of the field at the target position $\vec{r}$ using (3), with the transfer function (5). Then, the probability amplitude of finding the photon in $t, \vec{r}'$ is

$$\langle 0|A^+(t, \vec{r}')|\psi\rangle = \quad (9)$$
$$\int d^2\vec{r}_t \, d^3\vec{\kappa} \, h_\omega(\vec{r}_t\,', \vec{r}_t, d) \, e^{-i(\omega_k t - \vec{\kappa}\cdot\vec{r})} \langle 0|a(\vec{\kappa})\times$$
$$\int d^3\vec{\kappa}' \tilde{G}_{\omega'}(\vec{\kappa}') \, a^\dagger(\vec{\kappa}')|0\rangle =$$
$$\int d^2\vec{r}_t \, d^3\vec{\kappa} \, h_\omega(\vec{r}_t\,', \vec{r}_t, d) \, e^{-i(\omega t - \vec{\kappa}\cdot\vec{r})} \, \tilde{G}(\vec{\kappa}) \propto$$
$$\int d^2\vec{r}_t \, d\omega \, h_\omega(\vec{r}_t\,', \vec{r}_t, d) \, e^{-i\omega t} \, \varphi(\omega) \, G_\omega(\vec{r}_t, z),$$

---

[1] It would be more appropriate to use a coherent state (or a thermal state) to model a classical beam, but since the photons in coherent states are completely uncorrelated (Poissonian statistics), one can easily obtain the same arrival statistics as a coherent state $|\alpha\rangle$ (or its thermal mixtures) by considering what happens to $N = |\alpha|^2$ uncorrelated single photons. (Of course the photon number statistics will be different!)

where we used the commutator (A6) in the second equality, and in the third we used the far field condition $\kappa_z \sim \kappa \gg \kappa_x, \kappa_y$ to separate the integral over $\vec{\kappa}$ into a frequency and a transverse part $\vec{\kappa}_t$: $d^3\vec{\kappa} \propto d\omega d^2\vec{\kappa}_t$, so that the integral of $e^{i\vec{\kappa}\cdot\vec{r}}\psi(\vec{\kappa})$ over the transverse part of $\vec{\kappa}$ gives the spatial field at frequency ω, namely $U_\omega(\vec{r}) = \varphi(\omega)G_\omega(\vec{r}_t, z)$ with $\vec{r} = (\vec{r}_t, z)$ [compare Eqs. (A2) and (A4)]. Eq. (9) is compatible with what one would expect from the transfer function of the classical amplitudes: see Eq. (1) when the time evolution of the output field is added and both members are integrated over ω. We now use the fact that the free space transfer function (5) applied to a Gaussian beam (6) translates it forward by a factor d, the distance between target and receiver (namely, a Gaussian beam is transformed in a Gaussian beam thanks to the hypothesis that the target is a partially reflecting mirror larger than the beam waist). Then, (9) becomes

$$\langle 0|A^+(t, \vec{r}')|\psi\rangle = \int d\omega \, e^{-i\omega t} \varphi(\omega) G_\omega(\vec{r}_t\,', z + d) \,. \quad (10)$$

As expected, at the image plane at position z', it gives a pulse that is delayed by the transit time to the target. To see this, consider the expression (10) at the center of the image plane $\vec{r}_t\,' = 0$ where, from (6) we see that $G_\omega(\vec{r}_t\,' = 0, z) \propto e^{-i[\kappa z - \arctan(z/z_0)]}$, so that (10) becomes

$$\int d\omega \, e^{-i\omega[t + (z+d)/c] - i\arctan((z+d)/z_0)} \varphi(\omega)$$
$$= \tilde{\varphi}(t + (z+d)/c) \, e^{-i\arctan((z+d)/z_0)} \,, \quad (11)$$

where $\tilde{\varphi}$ is the Fourier transform of φ. Eq. (11) describes a pulse of spectral amplitude $\varphi(\omega)$ and temporal amplitude $\tilde{\varphi}(t)$ that is delayed by an amount $(z + d)/c$, where d is the distance between target and receiver and $z = d$ is the position of the target. By measuring the time of arrival of the photon, one can obtain twice the distance $2d$ to the target, as expected for a radar. The statistical error in this measurement is given by the width $\Delta\tau$ of $\tilde{\varphi}(\tau)$, proportional to the inverse of the bandwidth $\Delta\omega$ of $\varphi(\omega)$.

Now we could do the same calculation with a coherent state $|\alpha\rangle$ instead of a single photon state (8), with $|\alpha\rangle = \bigotimes_{\vec{\kappa}} |\alpha(\vec{\kappa})\rangle$ with $|\alpha(\vec{\kappa})\rangle$ eigenstates of $a(\vec{\kappa})$: $a(\vec{\kappa})|\alpha(\vec{\kappa})\rangle = \alpha(\vec{\kappa})|\alpha(\vec{\kappa})\rangle$. This calculation should give exactly the same outcome as a classical field amplitude $\alpha(\vec{\kappa})$, see Eq. (A4).

III.   QUANTUM RADAR/LIDAR PROTOCOL

We now show how one can obtain an increased localization precision by using frequency-entangled light. For simplicity of notation, we will consider only the case of $N = 2$ entangled photons. This can then be extended to arbitrary N.

For the N-photon state $|\psi_N\rangle$ of (A8), the probability of detecting them at $t_1, \vec{r}_1, \cdots, t_N, \vec{r}_N$ is [23]

$$p \propto |\langle 0|A^+(t_1, \vec{r}_1) \cdots A^+(t_N, \vec{r}_N)|\psi_N\rangle|^2 \,. \quad (12)$$

Consider the biphoton entangled state with wavefunction

$$\tilde{\psi}_2(\vec{r}, \vec{\rho}) \propto \int d\omega \, \varphi(\omega) \, G_\omega(\vec{r}, \vec{\rho}) \,, \qquad (13)$$

which gives the probability amplitude of finding the two photons at positions $\vec{r} = (\vec{r}_t, z)$ and $\vec{\rho} = (\vec{\rho}_t, \rho_z)$ (in the Heisenberg picture there is no time evolution), and where

$$G_\omega(\vec{r}, \vec{\rho}) \equiv \qquad (14)$$

$$\frac{1}{W(z+\rho_z)} e^{-\frac{\kappa(\vec{r}_t^2 + \vec{\rho}_t^2)}{2z_0 W^2}} e^{-i[\kappa(z+\rho_z) + \frac{\kappa(\vec{r}_t^2 + \vec{\rho}_t^2)}{2zR} - \arctan\frac{z+\rho_z}{z_0}]},$$

which represents two photons of identical frequency ω in a Gaussian beam, see Eq. (6). Except for the multiplicative term $1/W$ and the arctan term, G_ω is basically a product of two Gaussian beam single-photon amplitudes. So we can reuse the calculations above for the single-photon amplitude to find that the temporal amplitude at the center of the image plane $\vec{r}_t = \vec{\rho}_t = 0$ at the image plane position $z = \rho_z = z'$ is given by the analogous of (11):

$$\langle 0 | A^+(t_1, \vec{r}) A^+(t_2, \vec{\rho}) | \psi \rangle = \tilde{\varphi}(t_1 + t_2 + 2(z+d)/c) \, e^{i\theta} \,, \qquad (15)$$

with θ some irrelevant phase factor. From this, it is clear that the time of arrival sum $t_1 + t_2$ has an uncertainty $\Delta\tau$, the width of $\tilde{\varphi}$. Which means that the average time of arrival $(t_1 + t_2)/2$ is estimated to be the correct value $d + z = 2d$ with a statistical error $\Delta\tau/2$. Instead, from (11) we saw that, using a single photon state, one estimates the time of arrival with an uncertainty $\Delta\tau$, so the average time of arrival of two photons will be estimated with an uncertainty $\simeq \Delta\tau/\sqrt{2}$. The $\sqrt{2}$ enhancement in precision is the $\sqrt{N}$ gain that one expects from entanglement in quantum metrology.

The biphoton analysis done here can be straightforwardly extended to the case of N entangled photons in a Gaussian beam. Namely, a state with wavefunction

$$\psi(\vec{r}_1, \cdots, \vec{r}_N) \propto \int d\omega \varphi(\omega) G_\omega(\vec{r}_1, \cdots, \vec{r}_N) \,, \qquad (16)$$

where G_ω is a trivial generalization of (14). It gives a $\sqrt{N}$ enhancement in the average photon time of arrival, which translates into a $\sqrt{N}$ precision enhancement in the longitudinal localization for each point in the sky scanned by the N-photon Gaussian beam (16), when one measures the average arrival time $\sum_i t_i/N$.

IV. CONCLUSIONS

In conclusion we have presented a quantum radar protocol that uses entanglement in the frequency/wavelength degrees of freedom to provide an quantum enhancement equal to the square root $\sqrt{N}$

of the number N of entangled photons employed. We have shown in detail how the optical transfer function formalism can be employed in the fully quantum regime we analyze here.

Appendix A: Quantization of the EM field

In this appendix we review the usual theory for the quantization of the electromagnetic field. This is useful to set the notation we use in the paper, and also to keep track of the specific roles that all the radiation degrees of freedom play in our protocol. Specifically, it is useful to understand the peculiar role of the frequency degree of freedom of the radiation that our protocol hinges on.

1. Classical EM in the Coulomb gauge

Start from the Maxwell equations in vacuum in the Coulomb gauge for the scalar and vector potentials Φ and $\vec{A}$: $\nabla^2 \Phi(t, \vec{r}) = 0$, $\Box \vec{A}(t, \vec{r}) = \frac{\partial}{\partial t} \vec{\nabla} \Phi$, where $\vec{r} = (x, y, z)$ is the spatial position, and $\Box = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ is the d'Alamabertian with $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ in Cartesian coordinates. Conventionally, since we are interested only in the quantization of the electromagnetic waves, one chooses the specific solution $\Phi = 0$, which implies $\Box \vec{A}(t, \vec{r}) = 0$. For simplicity of notation we will consider scalar fields $A(t, \vec{r})$ from now on: the vectorial part can be added by introducing two independent components, connected to the two polarizations of the em field (there are two polarizations because, in the Coulomb gauge, the potential $\vec{A}$ is transverse: $\vec{\nabla} \cdot \vec{A} = 0$). We separate the temporal and spatial degrees of freedom by taking a Fourier transform over time:

$$A(t, \vec{r}) = \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \, U_\omega(\vec{r}) \,, \qquad (A1)$$

where U_ω is the component at frequency ω. Since A is real, we must have $U_{-\omega} = U_\omega^*$, this condition can be enforced automatically if we separate the integral into a sum of two and change variable in the second:

$$A(t, \vec{r}) = \int_0^\infty d\omega [e^{-i\omega t} U_\omega(\vec{r}) + e^{+i\omega t} U_\omega^*(\vec{r})] \,. \qquad (A2)$$

For each component at frequency ω, it is clear from (A1) that $\Box A = 0$ becomes the Helmholtz equation

$$\nabla^2 U_\omega(\vec{r}) = \frac{\omega^2}{c^2} U_\omega(\vec{r}). \qquad (A3)$$

A convenient[2] solution is in terms of plane waves $U_\omega(\vec{r}) = \alpha(\vec{\kappa}) e^{\pm i \vec{\kappa} \cdot \vec{r}}$. The real and imaginary part (or the modulus

---

[2] Because of the linearity of the Helmholtz equation, any solution $U_\omega(\vec{r})$ can be expressed as a sum of plane waves $U_\omega(\vec{r}) = \int d^3\kappa \, \alpha(\vec{\kappa}) \, e^{i\vec{\kappa} \cdot \vec{r}}$.

and phase) of α are the two integration constants, and the wave direction $\vec{\kappa}/\kappa$ parametrizes all the solutions, while we must choose $|\vec{\kappa}|$ such that $\omega = \vec{c} \cdot \vec{\kappa} = \pm|\vec{\kappa}|c$. The sign parametrizes two classes of solutions: we must choose $+$ if the wave vector $\vec{\kappa}$ is parallel to the wave velocity $\vec{c}$, or $-$ if it is antiparallel. The first case refers to the retarded waves, the second to the advanced waves ([24], sec. 6.4). We usually choose past boundary conditions, so we only use retarded waves[3]. Summarizing, the vacuum solution of the Maxwell equations is of the form

$$A(t, \vec{r}) = \int_{\mathbb{R}^3} d^3\vec{\kappa} [\alpha(\vec{k}) \, e^{-i(\omega_\kappa t - \vec{\kappa} \cdot \vec{r})} + \alpha^*(\vec{k}) \, e^{i(\omega_\kappa t - \vec{\kappa} \cdot \vec{r})}] \, ,$$
$$(A4)$$

where $\alpha(\vec{\kappa})$ is the positive-frequency amplitude[4] and $\omega_\kappa = |\vec{\kappa}|c$, so that the integral over $\vec{\kappa}$ takes care of the integral over ω in (A2) (its modulus) and of the integral over the directions $\vec{\kappa}/\kappa$ that enumerate all plane waves.

2. Quantum em: observables

The energy of the electromagnetic field is $H = \frac{\epsilon_0}{2} \int d^3\vec{r} [E^2(t, \vec{r}) + c^2 B^2(t, \vec{r})]$, where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. In terms of the amplitudes $\alpha(\vec{\kappa})$, one can show that, in the Coulomb gauge, the energy is

$$H = \frac{1}{2} \int d^3\vec{\kappa} \left(P_{\vec{\kappa}}^2 + \omega_\kappa^2 X_{\vec{\kappa}}^2 \right) \, , \qquad (A5)$$

with $P_{\vec{\kappa}} \propto i(\alpha_{\vec{\kappa}}^* - \alpha_{\vec{\kappa}})$ and $X_{\vec{\kappa}} \propto (\alpha_{\vec{\kappa}}^* + \alpha_{\vec{\kappa}})$. Eq. (A5) is the energy of a collection of independent (noninteracting) harmonic oscillators (one for each value of $\vec{\kappa}$), so we can quantize by considering X and P as "position" and "momentum" operators, promoting the amplitudes α to operators a. Namely, we impose $[X_{\vec{\kappa}}, P_{\vec{\kappa}'}] = i\delta(\vec{\kappa} - \vec{\kappa}')$, where the delta shows that they are independent oscillators for each $\vec{\kappa}$. From the definitions of X and P, this implies the commutators

$$[a(\vec{\kappa}), a^\dagger(\vec{\kappa}')] = \delta(\vec{\kappa} - \vec{\kappa}') \, , [a(\vec{\kappa}), a(\vec{\kappa}')] = 0 \, . \qquad (A6)$$

---

[3] If we were to choose future boundary conditions, we would need to consider only advanced waves, and if we were to choose mixed future-and-past boundary conditions [25], we would have to keep both solutions, which, incidentally, is a real problem in quantum field theory, as this leads to a non-Hamiltonian evolution of the electromagnetic field! Advanced waves can be seen as propagating negative energy in the forward time direction or positive energy in the negative time direction.

[4] Note that the Maxwell equations are solved also by negative frequency plane waves of the type $\alpha_-(\vec{\kappa}) \, e^{-i(\omega_\kappa t + \vec{\kappa} \cdot \vec{r})}$ but, as discussed above, we will ignore these solutions, by choosing past boundary conditions as is done usually.

The quantization of the general solution of the Maxwell equations in the Coulomb gauge $\Box A = 0$, is then

$$A(t, \vec{r}) = \int d^3\vec{\kappa} \, [a(\vec{k}) \, e^{-i(\omega_\kappa t - \vec{\kappa} \cdot \vec{r})} + a^\dagger(\vec{k}) \, e^{i(\omega_\kappa t - \vec{\kappa} \cdot \vec{r})}] \, ,$$
$$(A7)$$

namely Eq. (A4) quantized. Importantly, since we are introducing the time evolution in the operators, we are working in the Heisenberg picture (or in the interaction picture with the free-field Hamiltonian to evolve the operators). We are working with the vector potential field A, but the electric field and magnetic fields are trivially obtained from it: $\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi$, $\vec{B} = \vec{\nabla} \times \vec{A}$, which give expressions very similar to (A7), except for the fact that the derivatives introduce a minus sign between the two terms of the right-hand-side (as $\vec{A}$, also $\vec{E}$ and $\vec{B}$ have two independent components since they are transverse: $\vec{\nabla} \cdot \vec{B} = 0$ and, in vacuum, $\vec{\nabla} \cdot \vec{E} = 0$).

The intensity of the field is proportional to the time averaged square $\langle E^2 \rangle_t$. This is basically equal to the average photon number in the field. Indeed, from (A4) and the fact that $\vec{E} = -\frac{\partial}{\partial t}\vec{A}$ we have that, classically, $E^2 = -\omega(\alpha^2 e^{i\phi} + (\alpha^*)^2 e^{-i\phi} - 2|\alpha|^2)$ for a classical field with only a single $\vec{\kappa}$, with $\Phi = \omega t - \vec{\kappa} \cdot \vec{r}$. This almost matches the quantum result one would get for a coherent state $|\alpha\rangle$ for which $a|\alpha\rangle = \alpha|\alpha\rangle$ (which represents a classical field). A coherent state has $E^2 = -\omega(\alpha^2 e^{i\phi} + (\alpha^*)^2 e^{-i\phi} - 2|\alpha|^2 - 1)$, where the -1 term comes from the commutator $[a, a^\dagger] = 1$ (valid for the quantization of a field with single $\vec{\kappa}$ vector). The time average removes the terms with the phase, leaving only the average photon number for a coherent state, i.e. $\langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2$. So while E^2 does not coincide with the average photon number, the time-averaged E^2 essentially does for classical fields. Similar considerations apply also to states with fixed photon number we consider below, where $\langle a^2 \rangle = 0$.

3. Quantum em: states

In the classical case, we can choose a specific form of the $\alpha(\vec{\kappa})$ to obtain a specific solution of the Maxwell equations (which can be done by choosing appropriate boundary conditions for the field). In the quantum case, the $\alpha \to a(\vec{\kappa})$ are operators. There are two ways to assign a value to them: (i) have them act on eigenstates of the field (which implies that the field is in a state where there are no quantum fluctuations of the field). From the form of Eq. (A7), it is clear that the eigenstates of the field are quadrature eigenstates for each $\vec{\kappa}$, where the quadrature is $Q_\varphi \equiv (a \, e^{-i\varphi} + a^\dagger e^{i\varphi})/\sqrt{2}$. These eigenstates are unphysical as they are infinitely squeezed states with infinite average energy $\hbar\omega_\kappa \langle a^\dagger(\kappa) a(\kappa) \rangle$. (ii) we can calculate the field expectation value on an arbitrary state $|\psi\rangle$ of the field (which implies that we can calculate the average field, because there are quantum fluctuations: measuring the field multiple times, we would get different results).

How do we choose $|\psi\rangle$? It is the state of the em degrees of freedom. Since these are given by a harmonic oscillator for each $\vec{\kappa}$, the Hilbert space is a Fock space for each $\vec{\kappa}$, so the most general state is given by

$$|\psi\rangle = \sum_n \gamma_n |\psi_n\rangle, \text{ with} \tag{A8}$$

$$|\psi_n\rangle = \int d^3\vec{\kappa}_1 \cdots d^3\vec{\kappa}_n \, \psi_n(\vec{\kappa}_1, \cdots, \vec{\kappa}_n) a^\dagger(\vec{\kappa}_1) \cdots a^\dagger(\vec{\kappa}_n)|0\rangle,$$

where γ_n is the probability amplitude to have n photons, ψ_n is the joint probability amplitude that they are in modes $\vec{\kappa}_1, \cdots, \vec{\kappa}_n$, and $|0\rangle$ is the vacuum. In this paper we only consider states with a single component $\gamma_N = 1$. In the Heisenberg picture, the state does not evolve in time. The wavefunction normalization condition is

$$\int d^3\vec{\kappa}_1 \cdots d^3\vec{\kappa}_n \, |\psi_n(\vec{\kappa}_1, \cdots, \vec{\kappa}_n)|^2 = 1 \; \forall n \; . \tag{A9}$$

More rigorously, the state (A8) refers only to the situation in which all the photons are in different modes (namely, the ψ_n does not contain Dirac δs over the $\vec{\kappa}_i$). The most general situation is to have the n photons distributed into $m \leqslant n$ modes ($m = 1$ if all the n photons are in one mode, $m = n$ if there is one photon per mode, as above). The indistinguishable nature of the photons implies that only m of the $\binom{m}{n}$ possibilities can be tracked, namely we can only know that n_i photons are in mode $\vec{\kappa}_i$ for $i = 1, \cdots, m$. In this case, the wavefunction is $\psi_n(n_1, \vec{\kappa}_1, \cdots, n_m, \vec{\kappa}_m)/\sqrt{n_1! \cdots n_m!}$ with $\sum_i n_i = n$, where the factorials appear because the n_i photon Fock state in a mode is given by $|n_i\rangle = (a^\dagger)^{n_i}|0\rangle/\sqrt{n_i!}$ and where ψ_n is the joint probability amplitude that the n photons are partitioned as $\{n_i\}$ *and* that their wave vectors are $\{\vec{\kappa}_i\}$.

For single photon states, only the term $n = 1$ of (A8) survives. The spatial dependence of the wavefunction can be obtained by taking the Fourier transform $\tilde{\psi}$ of $\psi \equiv \psi_1$:

$$\int d^3\vec{\kappa} \, \psi(\vec{\kappa}) \, a^\dagger(\vec{\kappa})|0\rangle = \int d^3\vec{r} \, \tilde{\psi}(\vec{r}) \, a^\dagger(\vec{r})|0\rangle \; , \tag{A10}$$

where $a(\vec{r}) \propto \int d^3\vec{\kappa} a(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{r}}$ is the annihilator of a photon at position $\vec{r}$, so that $\tilde{\psi}(\vec{r})$ is the probability that the photon is in $\vec{r}$. (Note that, except in the limit discussed in the next subsection, this is *not* in general equal to the probability amplitude of measuring the photon at position $\vec{r}$, since there is a difference between the position of the photon and of its energy, a well known problem in quantum field theory, e.g. [23, 26, 27]. Indeed, as is clear from the above analysis, the photon is obtained from the quantization of the vector potential A, which is a gauge-dependent quantity, whereas its energy is, clearly, a gauge-independent quantity.)

We can choose $\psi(\vec{\kappa}) = \mathcal{N}\alpha(\vec{\kappa})$, where $\alpha(\vec{\kappa})$ is the Fourier transform of the classical solution $A(t, \vec{r})$ of (A4)

and $\mathcal{N}$ is a normalization for Eq. (A9). Indeed, as discussed below, $|A(t, \vec{r})|^2$ is the light intensity at position $t, \vec{r}$, so it is proportional to the probability of finding the photon at such position, so A is the probability amplitude, and its Fourier transform $\alpha(\vec{\kappa})$ is the probability amplitude in the $\vec{\kappa}$ space.

4. Quantum em: photodetection

It can be shown that for photodetectors with efficiency η, sufficiently small temporal resolution τ, and spatial resolution σ, the probability of a photodetection at spacetime position $(t, \vec{r})$ is [23] $p(t, \vec{r}) \propto \eta\tau\sigma\langle\psi|[A^+(t,\vec{r})]^\dagger A^+(t,\vec{r})|\psi\rangle$. In the case in which the system state $|\psi\rangle$ contains a single photon, we can use the fact that a is the photon annihilator to simplify it to $p \propto |\langle 0|A^+|\psi\rangle|^2$. To show this, consider $|\psi\rangle = \int d^3\vec{\kappa}'\psi(\vec{\kappa}')a^\dagger(\vec{\kappa}')|0\rangle$, with $\psi(\vec{\kappa}')$ the probability amplitude that the photon has wave vector $\vec{\kappa}'$ (so that its Fourier transform can be interpreted as the probability amplitude that the photon is in position $\vec{r}$). Then

$$A^+|\psi\rangle = \int d^3\vec{\kappa} \, d^3\vec{\kappa}' \psi(\vec{\kappa}') a(\vec{\kappa}) e^{-i(\omega_\kappa t - \vec{\kappa}\cdot\vec{r})} a^\dagger(\vec{\kappa}')|0\rangle =$$

$$\tilde{\psi}(\vec{r} - \vec{c}t)|0\rangle \; , \tag{A11}$$

where $\vec{c}$ is the speed of light with the direction $\vec{\kappa}/\kappa$ of the beam. Eq. (A11) follows from the commutator (A6) and the fact that $a|0\rangle = 0$, and where $\tilde{\psi}$ is the Fourier transform of ψ. [Note the use of the Heisenberg picture: the time evolution is only in the operator A^+, not in the state.]

Acknowledgments

[1] M. Lanzagorta, J. Uhlmann, Quantum radar. San Rafael, California (USA): Morgan and Claypool,66 (2012).

[2] J. F. Smith III, Quantum interferometer and radar theory based on N00N, M and M or linear combinations of entangled states, in Proc. Quantum information and computation VIII, New York, 770201 (2010)

[3] M. Malik, O. S. Magana-Loaiza, and R. W. Boyd, Quantum-secured imaging, Appl. Phy. Lett. **101**, 241103 (2012)

[4] D. Luong, C. W. Sandbo Chang, A. M. Vadiraj, A. Damini, C. M. Wilson, B. Balaji, Receiver Operating Characteristics for a Prototype Quantum Two-Mode Squeezing Radar, IEEE Trans. Aerosp. Electr. Sys. **56**, 3 (2020) and arXiv:1903.00101 (2019).

[5] S. Lloyd, Enhanced Sensitivity of Photodetection via Quantum Illumination, Science **321**, 1463 (2008).

[6] S.-H. Tan, B.I. Erkmen, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, S. Pirandola, J. H. Shapiro, Quantum Illumination with Gaussian States, Phys. Rev. Lett. **101**, 253601 (2008).

[7] L. Maccone, C. Ren, "Quantum Radar", Phys. Rev. Lett. **124**, 200503 (2020).

[8] L. Maccone, C. Ren, Erratum, submitted (2023).

[9] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced positioning and clock synchronization, Nature **412**, 417 (2001).

[10] S.M. Barnett, C. Fabre, A. Maitre, Ultimate quantum limits for resolution of beam displacements, Eur. Phys. J. D **22**, 513 (2003).

[11] S. L. Braunstein, C. M. Caves, G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. **247**, 135-173 (1996); S. L. Braunstein, C. M. Caves, Statistical distance and the geometry of quantum states. Phys. Rev. Lett. **72**, 3439 (1994).

[12] V. Giovannetti, S. Lloyd, L. Maccone, Quantum metrology, Phys. Rev. Lett. **96**, 010401 (2006).

[13] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: beating the standard quantum limit, Science **306**, 1330 (2004); V. Giovannetti, S. Lloyd, L. Maccone, Advances in quantum metrology, Nature Phot. **5**, 222 (2011).

[14] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. **7**, 125 (2009).

[15] R. Demkowicz-Dobrzaňski, M. Jarzyna, J. Kodyňski, Quantum Limits in Optical Interferometry, Progress in Optics, **60**, 345 (2015) also at arXiv:1405.7703 (2014).

[16] R. Demkowicz-Dobrzanski, J. Koldynski, M. Guta, The elusive Heisenberg limit in quantum-enhanced metrology, Nature Comm. **3**, 1063 (2012).

[17] B. M. Escher, R. L. de Matos Filho, L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nature Phys. **7**, 406 (2011).

[18] E. M. Kessler, I. Lovchinsky, A. O. Sushkov, and M. D. Lukin, Phys. Rev. Lett. 112, 150802 (2014); W. Dur, M. Skotiniotis, F. Fröwis, and B. Kraus, Phys. Rev. Lett. **112**, 080801 (2014); G. Arrad, Y. Vinkler, D. Aharonov and A. Retzker, Phys. Rev. Lett. **112**, 150801 (2014).

[19] R. Demkowicz-Dobrzański, L. Maccone, Using entanglement against noise in quantum metrology, Phys. Rev. Lett. **113**, 250801 (2014).

[20] V. Giovannetti, S. Lloyd, and L. Maccone, Positioning and clock synchronization through entanglement, Phys. Rev. A **65**, 022309 (2002).

[21] B.E.A. Saleh, M.C. Teich, Fundamentals of Photonics (John Wiley & Sons 1991).

[22] V. Giovannetti, S. Lloyd, L. Maccone, J. H. Shapiro, Sub-Rayleigh-diffraction-bound quantum imaging, Phys. Rev. A **79**, 013827 (2009).

[23] L. Mandel and E. Wolf *Optical Coherence and Quantum Optics* (Cambridge Univ. Press, Cambridge, 1995).

[24] J.D. Jackson, Classical Electrodynamics, Third Edition, (John Wiley & Sons, Inc., 1999)

[25] A. Einstein, On the present status of the radiation problem, Physicalische Zeitscrhift **10**, 185 (1909); English translation in The Collected Papers of A. Einstein, Vol. 2. (Princeton Univ. Press, 1989).

[26] M. G. Raymer and B. J. Smith; A. Muthukrishnan, M. O. Scully and M. S. Zubairy, in The Nature of Light: What is a Photon?, edited by C. Roychoudhuri, K. Creath, A. F. Kracklauer, Proceedings of SPIE Vol. 5866 (SPIE, Bellingham, WA, 2005); A. Muthukrishnan, M. O. Scully and M. S. Zubairy, Optics and Photonics News, 14, S-18 (2003).

[27] I. Bialynicki-Birula, Z. Bialynicka-Birula, Why photons cannot be sharply localized, Phys. Rev. A **79**, 032112 (2009).